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Relativistic Corrections to the Lagrangian for Interacting Charged Particles

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We give the Lagrangian of a system of moving charged particles up to the fourth approximation in $1/c$ neglecting dipole radiation of the system. In this case the appearance of the electromagnetic waves (quadrupole radiation) by moving charges occurs in the fifth approximation in $1/c$.

1. INTRODUCTION

We consider a system of n massive charged particles moving relative to each other under the influence of their mutual Coulomb force. The term "particle" means that the mass and charge are concentrated at a point in space. The Lagrangian for the case of charged particles depending on the coordinates $\bar{r}_i(t)$ describing the system and in the usual simplest case on the velocities $\bar{v}_i$, was first derived relativistically by Darwin (1920). Generally we can expand the Lagrangian into a power series with respect to $1/c$ (Breitenberger, 1968; Landau and Lifshitz, 1970) as

$$L=L^{(0)}+L^{(2)}+O(c^{-3}) \tag{1.1}$$

where the radiation of electromagnetic waves occurs in the third order in $1/c$. The symbol $O(c^{-3})$ means that the remaining terms of equation (1.1) are of order $1/c^3$ and so on. Nevertheless, in special cases in which the ratio of charge to mass is the same, a system of charged particles cannot radiate by dipole radiation. The dipole radiation is determined by the second derivative of the dipole moment of the system, that is

$$\bar{d}=\sum_{i=1}^{n} q_i\bar{r}_i=\sum_{i=1}^{n}\frac{q_i}{m_i}m_i\bar{r}_i=\text{const}\sum_{i=1}^{n} m_i\bar{r}_i=\text{const}\,\bar{R}\sum_{i=1}^{n} m_i \tag{1.2}$$

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where $\bar{R}$ is the position vector of the mass center of the system. From equation (1.2) one can take

$$\ddot{\bar{d}}=\text{const}\,\ddot{\bar{R}}\sum_{i=1}^{n} m_i=0 \tag{1.3}$$

which means that there is no dipole radiation for a closed system of particles, since the center of mass moves uniformily. Hence, we must expect to be able to find a Lagrangian accurate beyond the terms $O(c^{-2})$.

We can apply the nonrelativistic mechanics since the velocities of the particles are small, though not negligible, fractions of the velocity of light. To get the next approximation, we proceed in the following fashion to find the Lagrangian for any number of freely moving interacting particles up to the fifth order in $1/c$, that is, we write down

$$L=L^{(0)}+L^{(2)}+L^{(4)}+L^{(5)}+O(c^{-6}) \tag{1.4}$$

where $L^{(0)}$ are the lowest-order terms, $L^{(2)}$ are the second-order terms, $L^{(4)}$ are the fourth-order terms, and $L^{(5)}$ are the fifth-order terms. It is clear that there is a close analogy with the Lagrangian of a system of gravitating particles correctly to terms of the fourth and fifth order (Dionysiou, 1976, 1977). When the Lagrangian for any problem has been found, the transition to the Hamiltonian follows in the usual way, and then we have the expansion

$$H=H^{(0)}+H^{(2)}+H^{(4)}+H^{(5)}+O(c^{-6}) \tag{1.5}$$

Equation (1.5) above is often called the integral of energy of our system, since the function H does not involve the time explicitly.

2. THE LAGRANGIAN

Starting with the Lagrangian density (Goldstein, 1971)

$$L=\frac{E^2-B^2}{8\pi}-\rho\varphi+\frac{1}{c}\bar{j}\cdot\bar{A} \tag{2.1}$$

which leads to the field equations

$$\bar{\nabla}\cdot\bar{E}=4\pi\rho \tag{2.1a}$$

$$\bar{\nabla}\times\bar{B}=\frac{1}{c}\left(4\pi\bar{j}+\frac{\partial}{\partial t}\bar{E}\right) \tag{2.1b}$$

we get the volume integral of equation (2.1), which is the total Lagrangian for the electromagnetic field, that is

$$L=\int_V\left(\frac{E^2-B^2}{8\pi}-\rho\varphi+\frac{1}{c}\bar{j}\cdot\bar{A}\right)dV \tag{2.2}$$

where V is the volume of the charged particles system.

More generally, in special relativity theory, we can write

$$L=\int_V\left(\frac{E^2-B^2}{8\pi}-\rho\varphi+\frac{1}{c}\bar{j}\cdot\bar{A}\right)dV-\sum_{i=1}^{n}m_ic^2\left(1-\frac{v_i^2}{c^2}\right)^{1/2} \tag{2.3}$$

Now for a system which does not emit dipole radiation we get the required fourth- and fifth-order terms (Landau and Lifshitz, 1975) from

$$L^{(4,5)}=\frac{1}{2}\int_V\left(\frac{1}{c}\bar{j}\cdot\bar{A}-\rho\varphi\right)dV-\sum_{i=1}^{n}m_ic^2\left(1-\frac{v_i^2}{c^2}\right)^{1/2} \tag{2.4}$$

where the first term represents the mutual interactions between particles and field, and the second term the Lagrangian for the particles in the absence of the field. Hence, we try to obtain the Lagrangian terms of higher order than two (Darwin, 1920) and discuss the effects to which these terms lead (Golubenkov and Smorodinskii, 1956). Here, we note that since the Lagrangian is not associated with a definite mechanical system, it does not have to be given as the difference of a kinetic and potential energy.

Suppose a discrete set of point particles with vector positions $\bar{r}_i(t)$ and charges q_i, then the density function can be represented as

$$\rho(\bar{r},t)=\sum_{i=1}^{n}q_i\delta(\bar{r}-\bar{r}_i(t)) \tag{2.5}$$

where the mathematical function δ is the well-known δ function introduced by Dirac and defined by the conditions

$$\delta(\bar{r}-\bar{r}_i)=0 \qquad \text{when } \bar{r}\neq\bar{r}_i$$

$$\int_V\delta(\bar{r}-\bar{r}_i)\,dV=1, \qquad \int_V\rho\delta(\bar{r}-\bar{r}_i)\,dV=\sum_{i=1}^{n}q_i, \qquad \int_V f(\bar{r})\delta(\bar{r}-\bar{r}_i)\,dV=f(\bar{r}_i)$$

Then we take

$$\int_V|\bar{r}-\bar{r}_j|^{\lambda}\sum_{i=1}^{n}q_i\delta(\bar{r}-\bar{r}_i)\,d\bar{r}=\sum_{i=1}^{n}q_i|\bar{r}_i-\bar{r}_j|^{\lambda} \tag{2.6}$$

where $\lambda=0, \pm 1, \pm 2, \ldots$.

The current density $\bar{j}$ is connected with the charge density ρ by the relation

$$\bar{j}=\rho\bar{v} \tag{2.7}$$

where $\bar{v}$ is the velocity of the charges and it is a function of position in space.

By virtue of equations (2.5), (2.6), (2.7) the integral of equation (2.4) is equal to

$$\frac{1}{2}\int_V\left[\frac{1}{c}\sum_{i=1}^{n} q_i\delta(\bar{r}-\bar{r}_i)\bar{v}\cdot\bar{A}(\bar{r})-\sum_{i=1}^{n} q_i\delta(\bar{r}-\bar{r}_i)\varphi(\bar{r})\right]dV$$

$$=\frac{1}{2}\sum_{i=1}^{n} q_i\left[\frac{\bar{v}_i\cdot\bar{A}(\bar{r}_i)}{c}-\varphi(\bar{r}_i)\right] \tag{2.8}$$

Now, equation (2.4) can be written as

$$L^{(4,5)}=\frac{1}{2c}\sum_{i=1}^{n} q_i\bar{v}_i\cdot\bar{A}(\bar{r}_i)-\frac{1}{2}\sum_{i=1}^{n} q_i\varphi(\bar{r}_i)-\sum_{i=1}^{n} m_i c^2\left(1-\frac{v_i^2}{c^2}\right)^{1/2} \tag{2.9}$$

Equation (2.9) gives us the required part of the Lagrangian, which describes both the electromagnetic field on the one hand and the mechanical motion of the n particles on the other. Now going back to the third-order terms we have $L^{(3)}=0$ (Landau and Lifshitz, 1975). Also, we note that there is a close agreement with a purely gravitational case, where $L^{(3)}=0$ (Dionysiou, 1976, 1977). We consider the retarded potentials of equation (2.9) at the positions $\bar{r}_i$ since, when several particles are free to move, the force exerted by one of them on another depends on its position and motion at certain previous time, that is

$$\varphi(\bar{r}_i,t)=\int_V\frac{\rho_{(t-R/c)}}{R}dV,\ \bar{A}(\bar{r}_i,t)=\frac{1}{c}\int_V\frac{\bar{j}_{(t-R/c)}}{R}dV \tag{2.10}$$

where $R=|\bar{r}_i-\bar{r}|$ is the distance from the volume element $dV\equiv d\bar{r}$ to the "field point" $\bar{r}_i$ at which we determine the potentials.

If the motion of the charges is sufficiently slow and smooth (i.e., we impose a limitation on the speed of the charge ($v^2 \ll c^2$), on its acceleration and on higher time derivative), we can expand the functions $\rho_{(t-R/c)}$ and $\bar{j}_{(t-R/c)}$ in inverse powers of c as

$$\varphi=\int_V \frac{\rho}{R}\,dV-\frac{1}{c}\frac{\partial}{\partial t}\int_V \rho\,dV+\frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int_V R\rho\,dV-\frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int_V R^2\rho\,dV$$

$$+\frac{1}{24c^4}\frac{\partial^4}{\partial t^4}\int_V R^3\rho\,dV-\frac{1}{120c^5}\frac{\partial^5}{\partial t^5}\int_V R^4\rho\,dV+O(c^{-6}) \tag{2.11}$$

where the scalar potential φ takes the expansion form

$$\varphi=\varphi^{(0)}+\varphi^{(2)}+\varphi^{(3)}+\varphi^{(4)}+\varphi^{(5)}+O(c^{-6}) \tag{2.12}$$

In equation (2.11) the time differentiation can clearly be taken out from under the integral sign. In equation (2.12) we have $\varphi^{(1)}=0$, since

$$\frac{\partial}{\partial t}\int_V \rho\,dV=0 \tag{2.13}$$

where this integral means the total charge of the system and is therefore independent of time. Also, the expansion of the vector potential is given by

$$\bar{A}=\frac{1}{c}\int_V \frac{\bar{j}}{R}\,dV-\frac{1}{c^2}\frac{\partial}{\partial t}\int_V \bar{j}\,dV+\frac{1}{2c^3}\frac{\partial^2}{\partial t^2}\int_V \bar{j}R\,dV$$

$$-\frac{1}{6c^4}\frac{\partial^3}{\partial t^3}\int_V \bar{j}R^2\,dV+O(c^{-5}) \tag{2.14}$$

where the time differentiation can clearly be taken out from under the integral sign and it has the form

$$\bar{A}=\bar{A}^{(1)}+\bar{A}^{(2)}+\bar{A}^{(3)}+\bar{A}^{(4)}+O(c^{-5}) \tag{2.15}$$

Substituting the expressions (2.11) and (2.14) in the required approximation into equation (2.9) we can find the Lagrangian of the fourth and fifth order, which must be added to the known second-order Lagrangian

(Darwin, 1920; Landau and Lifshitz, 1975), for the whole system as

$$L^{(4,5)} = \frac{1}{c^4}\left(\frac{1}{4}\sum_{i=1}^{n} q_i \bar{v}_i \cdot \frac{\partial^2}{\partial t^2}\int_V \bar{j} R \, dV - \frac{1}{48}\sum_{i=1}^{n} q_i \frac{\partial^4}{\partial t^4}\int_V R^3 \rho \, dV + \frac{1}{16}\sum_{i=1}^{n} m_i v_i^6\right) + \frac{1}{c^5}\left(-\frac{1}{12}\sum_{i=1}^{n} q_i \bar{v}_i \cdot \frac{\partial^3}{\partial t^3}\int_V \bar{j} R^2 \, dV + \frac{1}{240}\sum_{i=1}^{n} q_i \frac{\partial^5}{\partial t^5}\int_V \rho R^4 \, dV\right) + O(c^{-6}) \tag{2.16}$$

where

$$\rho = \sum_{j=1}^{n} q_j \delta(\bar{r} - \bar{r}_j) \quad \text{and} \quad \bar{R} = \bar{r}_i - \bar{r}$$

For the fourth-order terms of the Lagrangian the corresponding scalar and vector potentials are

$$\varphi^{(4)} = \frac{1}{24c^4}\frac{\partial^4}{\partial t^4}\int_V R^3 \rho \, dV \tag{2.17}$$

$$\frac{1}{c}\bar{A}^{(3)} = \frac{1}{2c^4}\frac{\partial^2}{\partial t^2}\int_V \bar{j} \mathrm{P} \, dV \tag{2.18}$$

then since the scalar potential φ is incovenient in our problem because it involves not only the velocities $\bar{v}_i$ but also the accelerations $\dot{\bar{v}}_i$ of the charges, which produce the field, we impose the "gauge condition"

$$\varphi' = \varphi - \frac{1}{c}\frac{\partial X}{\partial t}, \qquad \bar{A}' = \bar{A} + \bar{\nabla} X \tag{2.19}$$

since the scalar and vector potentials are not independent quantities but are connected by this gauge condition (2.19), where X is an arbitrary function of space and time coordinates. Furthermore, the $\bar{\nabla}$ operator means differentiation with respect to the coordinates of the "field point" at which we seek the value of $\bar{A}'$. Let us now make use of the function

$$X = \frac{1}{24c^3}\frac{\partial^3}{\partial t^3}\int_V R^3 \rho \, dV$$

and equations (2.17), (2.18), (2.19), then it follows that

$$\varphi'^{(4)}=0$$

and

$$\frac{1}{c}\bar{A}'^{(3)}(\bar{r}_i)=\frac{1}{2c^4}\frac{\partial^2}{\partial t^2}\int_V \bar{j}R\,dV+\frac{1}{24c^4}\bar{\nabla}_{\bar{r}_i}\frac{\partial^3}{\partial t^3}\int_V R^3\rho\,dV \qquad (2.20)$$

Putting into equation (2.20), equations (2.5), (2.7) and integrating according to equation (2.6) we get that

$$\frac{1}{c}\bar{A}'^{(3)}(\bar{r}_i)=\frac{1}{2c^4}\sum_{j=1}^{n} q_j\frac{\partial^2}{\partial t^2}\left[R\bar{v}_j+\frac{1}{12}\frac{\partial}{\partial t}\left(\bar{\nabla}_{\bar{r}_i}R^3\right)\right] \qquad (2.21)$$

where we have put $\bar{\nabla}(\partial^3 R/\partial t^3)=(\partial^3/\partial t^3)\bar{\nabla}R$. After performing some of the differentiations in equation (2.21) i.e.,

$$\bar{\nabla}_{\bar{r}_i}R^3=3R^2\bar{\nabla}_{\bar{r}_i}R=3R^2\frac{\bar{r}_i-\bar{r}_j}{|\bar{r}_i-\bar{r}_j|}=3R^2\bar{\eta}$$

where $\bar{R}=\bar{r}_i-\bar{r}_j$, and $\bar{\eta}$ is the unit vector in the direction of $\bar{R}$. Now, since $R^2=\bar{R}^2$ it follows, differentiating with respect to $\bar{r}_j$, that

$$R\dot{R}=\bar{R}\cdot\dot{\bar{R}}=-\bar{R}\cdot\bar{v}_j$$

thus

$$\dot{\bar{\eta}}=\frac{\partial}{\partial t}\left(\frac{\bar{R}}{R}\right)=\frac{R\dot{\bar{R}}-\dot{R}\bar{R}}{R^2}=\frac{-\bar{v}_j+\bar{\eta}(\bar{\eta}\cdot\bar{v}_j)}{R}$$

then equation (2.21) can be written as

$$\frac{1}{c}\bar{A}'^{(3)}(\bar{r}_i)=\frac{1}{8c^4}\sum_{j=1}^{n} q_j\frac{\partial^2}{\partial t^2}\left[3R\bar{v}_j-R\bar{\eta}(\bar{\eta}\cdot\bar{v}_j)\right] \qquad (2.22)$$

where we have assumed the differentiation $\partial/\partial t$ is done for a fixed position of the "field point" $\bar{r}_i$, i.e., only with respect to $\bar{r}_j$; while the differentiation $\bar{\nabla}_{\bar{r}_i}$ is with respect to the coordinates of the "field point" $\bar{r}_i$. Defining the vector

$$\bar{F}_j=\frac{\partial}{\partial t}\left[3R\bar{v}_j-R\bar{\eta}(\bar{\eta}\cdot\bar{v}_j)\right],\qquad j=1,2,3,\ldots,n \qquad (2.23)$$

where $\partial/\partial t$ is with respect to $\bar{r}_j$ for a fixed position of $\bar{r}_i$; as a result we find from equations (2.22), (2.23) that

$$\frac{1}{c}\bar{A}'^{(3)}(\bar{r}_i)=\frac{1}{8c^4}\sum_{j=1}^{n} q_j\frac{\partial \bar{F}_j}{\partial t} \tag{2.24}$$

Now, in accordance with equations (2.16), (2.24) the expression for $L^{(4)}$ can be written in another form as

$$L^{(4)}=\frac{1}{16c^4}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j \bar{v}_i\cdot\frac{\partial \bar{F}_j}{\partial t}+\frac{1}{16c^4}\sum_{i=1}^{n} m_i v_i^6 \tag{2.25}$$

where $i\neq j$. It is easy to that the total time derivative of $\bar{v}_i\cdot\bar{F}_j$ is written by

$$\frac{d}{dt}\left(\bar{v}_i\cdot\bar{F}_j\right)=\frac{\partial}{\partial t}\left(\bar{v}_i\cdot\bar{F}_j\right)+\left(\bar{v}_i\cdot\bar{\nabla}_{\bar{r}_i}\right)\left(\bar{v}_i\cdot\bar{F}_j\right),\qquad i\neq j \tag{2.26}$$

From equations (2.25), (2.26) one easily obtains a Lagrangian $L^{(4)}$ completely describing (in this approximation) the motion of the charges, i.e.,

$$L^{(4)}=\frac{1}{16c^4}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i q_j\left[-\left(\bar{v}_i\cdot\bar{\nabla}_{\bar{r}_i}\right)\left(\bar{v}_i\cdot\bar{F}_j\right)-\dot{\bar{v}}_i\cdot\bar{F}_j\right]$$
$$+\frac{1}{16c^4}\sum_{i=1}^{n} m_i v_i^6,\qquad i\neq j \tag{2.27}$$

The term $(d/dt)(\bar{v}_i\cdot\bar{F}_j)$ can be dropped from the Lagrangian (2.27) as a total time derivative. Also, we define

$$\dot{\bar{v}}_i=-\frac{1}{m_i}\frac{\partial V}{\partial \bar{r}_i} \tag{2.28}$$

where

$$V=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{q_i q_j}{|\bar{r}_i-\bar{r}_j|},\qquad i\neq j \tag{2.28a}$$

that is to say, the accelerations is equation (2.27) can be expressed from the lowest approximation. The combination of equations (1.1) and (2.27) yields the complete relativistic Lagrangian up to the fourth order of $1/c$.

To calculate $L^{(5)}$ it is sufficient to know the potentials

$$\varphi^{(5)} = -\frac{1}{120c^5}\frac{\partial^5}{\partial t^5}\int_V R^4\rho\, dV \tag{2.29}$$

and

$$\frac{1}{c}\bar{A}^{(4)} = -\frac{1}{6c^5}\frac{\partial^3}{\partial t^3}\int_V R^2\bar{j}\, dV \tag{2.30}$$

from equations (2.11) and (2.14). Choosing the function

$$X = \frac{1}{120c^4}\frac{\partial^4}{\partial t^4}\int_V R^4\rho\, dV$$

then from equations (2.19) we can bring the potentials $\varphi^{(5)}$ and $\bar{A}^{(4)}$ to the equivalent forms

$$\varphi'^{(5)} = 0$$

and

$$\frac{1}{c}\bar{A}'^{(4)}(\bar{r}_i) = -\frac{1}{6c^5}\frac{\partial^3}{\partial t^3}\int_V R^2\bar{j}\, dV + \frac{1}{120c^5}\frac{\partial^4}{\partial t^4}\bar{\nabla}_{\bar{r}_i}\int_V R^4\rho\, dV \tag{2.31}$$

where we have put $\bar{\nabla}(\partial^4 R|\partial t^4) = (\partial^4|\partial t^4)\bar{\nabla}R$. It is clear that

$$\bar{\nabla}_{\bar{r}_i} R^4 = 4R^3\bar{\nabla}_{\bar{r}_i} R = 4R^3\frac{\bar{r}_i - \bar{r}}{|\bar{r}_i - \bar{r}|} = 4R^2\bar{R}$$

and $\dot{\bar{R}} = -\dot{\bar{r}} = -\bar{v}$ with respect to $\bar{r}$. We therefore have from the above relations and equation (2.16)

$$\begin{aligned} L^{(5)} &= \frac{1}{2}\sum_{i=1}^{n} q_i\bar{v}_i\cdot\left(-\frac{1}{6c^5}\frac{\partial^3}{\partial t^3}\int_V R^2\bar{j}\, dV + \frac{1}{30c^5}\frac{\partial^4}{\partial t^4}\int_V R^2\bar{R}\rho\, dV\right) \\ &= -\frac{1}{12c^5}\sum_{i=1}^{n} q_i\bar{v}_i\cdot\frac{\partial^3}{\partial t^3}\left[\sum_{j=1}^{n} q_j|\bar{r}_i - \bar{r}_j|^2\bar{v}_j - \frac{1}{5}\frac{\partial}{\partial t}\sum_{j=1}^{n} q_j|\bar{r}_i - \bar{r}_j|^2(\bar{r}_i - \bar{r}_j)\right] \end{aligned} \tag{2.32}$$

The second term appearing on the right side of the square bracket of the above equation gives

$$\sum_{j=1}^{n} q_j\frac{\partial}{\partial t}(R^2\bar{R}) = \sum_{j=1}^{n} q_j\frac{\partial}{\partial t}(R^3\bar{\eta}) \tag{2.33}$$

and since

$$\frac{\partial}{\partial t}(R^3\bar{\eta})=3R^2\dot{R}\bar{\eta}+R^3\dot{\bar{\eta}}=-3R^2\bar{\eta}\frac{\bar{R}\cdot\bar{v}_j}{R}+R^3\frac{-\bar{v}_j+\bar{\eta}(\bar{\eta}\cdot\bar{v}_j)}{R}$$

$$=-3R\bar{\eta}\left(\bar{R}\cdot\bar{v}_j\right)+R^2\left[-\bar{v}_j+\bar{\eta}\left(\bar{\eta}\cdot\bar{v}_j\right)\right]=-R^2\bar{v}_j-2R\bar{\eta}\left(\bar{R}\cdot\bar{v}_j\right)$$

differentiating for a fixed $\bar{r}_i$, it follows that

$$\frac{\partial}{\partial t}\left[\sum_{j=1}^{n}q_j|\bar{r}_i-\bar{r}_j|^2(\bar{r}_i-\bar{r}_j)\right]=-\sum_{j=1}^{n}q_j\left[R^2\bar{v}_j+2R\bar{\eta}\left(\bar{R}\cdot\bar{v}_j\right)\right] \tag{2.34}$$

Hence, from equations (2.32), (2.34) we take

$$L^{(5)}=-\frac{1}{30c^5}\sum_{i=1}^{n}\sum_{j=1}^{n}q_iq_j\bar{v}_i\cdot\frac{\partial^3}{\partial t^3}\left[3R^2\bar{v}_j+R\bar{\eta}\left(\bar{R}\cdot\bar{v}_j\right)\right],\qquad i\neq j \tag{2.35}$$

where $\partial/\partial t$ with respect to $\bar{r}_j$. Now, since equation (2.35) has the form

$$L^{(5)}=\frac{1}{2c}\sum_{i=1}^{n}q_i\bar{v}_i\cdot\bar{A}'^{(4)}(\bar{r}_i) \tag{2.36}$$

where the vector potential $\bar{A}'^{(4)}(\bar{r}_i)$ is independent of $\bar{v}_i$, one immediately sees that

$$\frac{\partial L^{(5)}}{\partial\bar{v}_i}=-\frac{1}{30c^5}\sum_{j=1}^{n}q_iq_j\frac{\partial^3}{\partial t^3}R^2\left[3\bar{v}_j+\bar{\eta}\left(\bar{\eta}\cdot\bar{v}_j\right)\right] \tag{2.37}$$

and therefore, one can thus obtain the Hamiltonian part $H^{(5)}$, i.e.,

$$H^{(5)}=\sum_{i=1}^{n}\frac{\partial L^{(5)}}{\partial\bar{v}_i}\cdot\bar{v}_i-L^{(5)}=0 \tag{2.38}$$

Here as a final result, we obtain the following expression for the Hamiltonian of the system (Dionysiou and Vaiopoulos, 1979):

$$H=H^{(0)}+H^{(2)}+H^{(4)}+O(c^{-6}) \tag{2.39}$$

Equation (2.39) is the integral of energy, since the function H does not involve the time explicitly, therefore

$$H=\text{const.}$$

It should be pointed out, if H is a constant of motion it is not always the integral of energy (Goldstein, 1971).

Following Landau and Lifshitz (1975), the total radiation, in which charge-to-mass ratio is the same for all the moving charges, is given by

$$\frac{1}{180c^5}\dddot{D}^2_{\alpha\beta}$$

i.e., the system emits electromagnetic waves by quadrupole radiation in unit time in all directions, where the tensor

$$D_{\alpha\beta}=\Sigma q\left(3x_\alpha x_\beta-\delta_{\alpha\beta}r^2\right), \qquad \alpha,\beta=1,2,3$$

is the quadrupole moment of the system. It should be noted here that we have a close agreement with a purely gravitational case of n-particle system (Dionysiou, 1977), i.e., equation (2.39). Also, the energy loss of the above gravitational system, when we average over the time, is given by

$$\frac{G}{45c^5}\dddot{D}^2_{\alpha\beta}$$

i.e., the system emits gravitational waves by quadrupole radiation (Dionysiou, 1979). Comparing the two above results we see that the gravitational radiation is very much smaller than the electromagnetic radiation because of the smallness of the gravitational constant G.

3. CONCLUSIONS

Work has been done on this problem for a system of two identical charged particles to terms of fourth order in the past (Golubenkov and Smorodinskii, 1956; Landau and Lifshitz, 1975). Equation (2.27) for $\eta=2$ gives us the known result. The fifth-order terms in the expansion of the field lead to certain additional forces acting on the charges, not contained in the Lagrangian, since there is quadrupole radiation of the system in this approximation. Therefore, we can define a Lagrangian completely describing the motion of charges only up to the fourth order in $1/c$.

Up to now, we know the combined Lagrangian (the influence of the gravitational field on the electromagnetic one and vice versa) of second-order approximation (Bażański, 1956, 1957; Barker and O'Connell, 1977, 1978). We thus see the necessity of establishing a theory for the Lagrangian of the gravitational and electromagnetic field up to the fourth order (Dionysiou, 1980), since the Lagrangian yields a convenient starting point in discussing particle dynamics. Of course, if one uses only the case of the electrovacuum then the appropriate Lagrangian is simply the sum of the Lagrangians of the gravitational and the electromagnetic field (Papapetrou, 1974).

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